ALGEBRAIC MULTILEVEL PRECONDITIONING IN ISOGEOMETRIC ANALYSIS: CONSTRUCTION AND NUMERICAL STUDIES

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ABSTRACT. We present algebraic multilevel iteration (AMLI) methods for isogeometric discretization of scalar second order elliptic problems. The construction of coarse grid operators and hierarchical complementary operators are given. Moreover, for a uniform mesh on a unit interval, the explicit representation of B-spline basis functions for a fixed mesh size h is given for $p = 2, 3, 4$ and for C^0- and C^{p-1}-continuity. The presented methods show h- and (almost) p-independent convergence rates. Supporting numerical results for convergence factor and iterations count for AMLI cycles (V-, linear W-, and nonlinear W-) are provided. Numerical tests are performed, in two-dimensions on square domain and quarter annulus, and in three-dimensions on quarter thick ring.

1. INTRODUCTION

The IsoGeometric Analysis (IGA), proposed by Hughes et al. in [27], has received great attention in the computational mechanics community. The concept has the capability of leading to large steps forward in computational efficiency since effectively, the process of re-meshing is either eliminated or greatly suppressed. The geometry description of the underlying domain is adopted from a Computer Aided Design (CAD) parametrization which is usually based on Non-Uniform Rational B-splines (NURBS), and the same basis functions are employed to approximate the physical solution. Since its introduction, IGA techniques have been studied and applied in diverse fields, see e.g., [1, 8, 9, 16, 17, 20, 26, 28]. Moreover, some theoretical aspects such as approximation properties, condition number estimates have been studied, see [7, 10, 14, 25]. The isogeometric methods, depending on various choices of basis functions, have shown several advantages over standard Finite Element Methods (FEM). For example, some common geometries arising in engineering and applied sciences, such as circles or ellipses, are exactly represented, and complicated geometries are represented more accurately than traditional polynomial based approaches. When we compare NURBS based isogeometric analysis with standard Lagrange polynomials based finite element analysis, it leads to qualitatively more accurate results [19]. Another limitation of finite element analysis is that it suits well for C^0 continuous interpolation, but for C^1 or higher order interpolation finite elements are complicated and expensive to construct. IGA offers C^{p-k}-continuous interpolation for p-degree basis functions with knot multiplicity k. Moreover, the ease in building spaces with high inter-element regularity allows for rather small problem sizes (in terms of degrees of freedom) with respect to standard finite element methods with the same approximation properties. This implies that, in general, for same approximation properties IGA stiffness and mass matrices are smaller than the corresponding finite element ones. Nevertheless, IGA discrete problems may still be very large in realistic problems of interest, and their condition numbers grow quickly with the inverse of mesh size h and the polynomial degree p. A detailed study of condition number estimates for stiffness matrix and mass matrix arising in isogeometric discretizations is given in [25]. For the h-refinement, the condition number of the stiffness matrix is bounded above and below by a constant times h^{-2}, and the condition number of the mass matrix is uniformly bounded. For the p-refinement, the condition number is bounded above by $p^{2d}4^{pd}$ and $p^{2(d-1)}4^{pd}$ for the stiffness matrix and the mass matrix, respectively. As a consequence, the cost of solving the linear system of equations arising from the isogeometric discretization, particularly using iterative solvers, becomes an important issue. Therefore, there is currently a growing interest in the design of efficient preconditioners for IGA discrete problems, in both the mathematical and the engineering communities. Multigrid methods for IGA have been introduced for two and three

dimensional elliptic problems by the authors in [24], and tearing and interconnecting methods for isogeometric analysis is discussed in [29]. Other recent work on solvers for IGA studied overlapping additive Schwarz methods [11, 13] and balancing domain decomposition by constraints methods [12]. Some issues arising in using direct solvers have been investigated in [18]. The results, we presented in [24], show optimal convergence rate with respect to the mesh parameter h. However, for discretizations based on higher degree polynomials, the convergence rate are quickly deteriorated. In this paper we discuss the construction of linear solvers which provide not only h-independent convergence rates but also exhibit (almost) independence on p. The presented optimal order solvers are based on algebraic multilevel iteration (AMLI) methods.

AMLI methods were introduced by Axelsson and Vassilevski in a series of papers [3, 4, 5, 6]. The AMLI methods, which are recursive extensions of two-level multigrid methods for FEM [2], have been extensively analyzed in the context of conforming and nonconforming FEM (including discontinuous Galerkin methods). For a detailed systematic exposition of AMLI methods, see the monograph [31, 38]. To reduce the overall complexity of AMLI methods (to achieve optimal computational complexity), various stabilization techniques can be used. In the original work [3, 4], the stabilization was achieved by employing properly shifted and scaled Chebyshev polynomials. This approach requires the computation of polynomial coefficients which depends on the bounds of the eigenvalues of the preconditioned system. Alternatively, some inner iterations at coarse levels can be used to stabilize the outer iterations, which lead to parameter-free AMLI methods [5, 6, 30, 34]. These methods utilize a sequence of coarse-grid problems that are obtained from repeated application of a natural (and simple) hierarchical basis transformation, which is computationally advantageous. Moreover, the underlying technique of these methods often requires only a few minor adjustments (mainly two-level hierarchical basis transformation) even if the underlying problem changes significantly.

In this article we consider the scalar second order elliptic equation as our model problem. Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be an open, bounded and connected Lipschitz domain with Dirichlet boundary $\partial\Omega$. We consider

$$(1) \qquad -\nabla \cdot (\mathcal{A}\nabla u) = f \quad \text{in } \Omega, \qquad u = u^D \quad \text{on } \partial\Omega,$$

where $\mathcal{A}(x)$ is a uniformly bounded function for $x \in \Omega$. Let $V^0 \subset H^1(\Omega)$ denote the space of test functions which vanish on $\partial\Omega$, and $V^D = V^0 + u^D \subset H^1(\Omega)$ denote the set which contains the functions fulfilling the Dirichlet boundary condition on $\partial\Omega$. By $V_h^0 \subset V^0$ and $V_h^D \subset V^D$ we denote the finite-dimensional spaces of the B-spline (NURBS) basis functions.

Introducing the bilinear form $a(\cdot, \cdot)$ and the linear form $f(\cdot)$ as

$$(2) \qquad a(u, v) = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v \, dx, \quad f(v) = \int_\Omega f \, v \, dx,$$

the Galerkin formulation of this problem reads:
Find $u_h \in V_h^D$ such that

$$(3) \qquad a(u_h, v_h) = f(v_h) \quad \text{for all } v_h \in V_h^0.$$

It is well known that (3) is a well-posed problem and has a unique solution. By approximating u_h and v_h using B-splines (NURBS) the variational formulation (3) is transformed in to a set of linear algebraic equations

$$(4) \qquad A\mathbf{u} = \mathbf{f},$$

where A denotes the stiffness matrix obtained from the bilinear form $a(\cdot, \cdot)$, i.e.

$$A = (a_{i,j}) = (a(N_i, N_j)), \quad i, j = 1, 2, 3, \dots, n_h,$$

$\mathbf{u}$ denotes the vector of unknown degrees of freedom (DOF), and $\mathbf{f}$ denotes the right hand side (RHS) vector from the known data of the problem. Clearly, A is a real symmetric positive definite matrix.

The rest of the paper is organized as follows. In Section 2 we briefly review the basics of B-splines and NURBS. An explicit representation of basis functions is also given in this section. The description of multilevel representation of B-splines (NURBS) is given in Section 3. A brief description of AMLI methods is given in Section 4. We then construct the isogeometric hierarchical spaces in Section 5. Numerical study of space splitting techniques is discussed in Section 6. The results of several numerical

experiments in two- and three-dimensions are presented in Section 7. Finally, some conclusions are drawn in Section 8.

2. B-SPLINES AND NURBS

2.1. **B-splines.** We first recall the definition of B-splines, see e.g. [35, 36].

Definition 1. *Let $\Xi_1 = \{\xi_i : i = 1, ..., n + p + 1\}$ be a non-decreasing sequence of real numbers, called the knot vector, where ξ_i is the ith knot, p is the polynomial degree, and n is the number of basis function. With a knot vector in hand, the B-spline basis functions, denoted by $N_i^p(\xi)$, are (recursively) defined starting with a piecewise constant*

$$(5a) \qquad B_i^0(\xi) = \begin{cases} 1 & \text{if } \xi \in [\xi_i, \xi_{i+1}), \\ 0 & \text{otherwise,} \end{cases}$$

$$(5b) \qquad B_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1}^{p-1}(\xi),$$

where $0 \le i \le n, p \ge 1$, and $\frac{0}{0}$ is considered as zero.

The above expression is usually referred as the *Cox-de Boor recursion formula*, see e.g. [15]. For a B-spline basis function of degree p, an interior knot can be repeated at most p times, and the boundary knots can be repeated at most $p + 1$ times. A knot vector for which the two boundary knots are repeated $p + 1$ times is said to be open. In this case, the basis functions are interpolatory at the first and the last knot. Important properties of the B-spline basis functions include nonnegativity, partition of unity, local support and C^{p-k}-continuity.

Definition 2. *A B-spline curve $C(\xi)$, is defined by*

$$(6) \qquad C(\xi) = \sum_{i=1}^{n} P_i B_i^p(\xi)$$

where $\{P_i : i = 1, ..., n\}$ are the control points and B_i^p are B-spline basis functions defined in (5).

The previous definitions are easily generalized to the higher dimensional cases by means of tensor product. Using tensor product of one-dimensional B-spline functions, a B-spline surface $S(\xi, \eta)$ is defined as follows:

$$(7) \qquad S(\xi, \eta) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} B_{i,j}^{p_1,p_2}(\xi, \eta) P_{i,j},$$

where $P_{i,j}, i = 1, 2, \ldots, n_1, j = 1, 2, \ldots, n_2$, denote the control points, $B_{i,j}^{p_1,p_2}$ is the tensor product of B-spline basis functions $B_i^{p_1}$ and $B_j^{p_2}$, and $\Xi_1 = \{\xi_1, \xi_2, \ldots, \xi_{n_1+p_1+1}\}$ and $\Xi_2 = \{\eta_1, \eta_2, \ldots, \eta_{n_2+p_2+1}\}$ are the corresponding knot vectors. Similarly, B-spline solids can be defined by a three-dimensional tensor product.

2.2. **NURBS.** While B-splines (polynomials) are flexible and have many nice properties for curve design, they are also incapable of representing curves such as circles, ellipses, etc. exactly. Such limitations are overcome by NURBS functions. Rational representation of conics originates from projective geometry and requires additional parameters called weights, which we shall denote by w. Let $\{P_i^w\}$ be a set of control points for a projective B-spline curve in $\mathbb{R}^3$. For the desired NURBS curve in $\mathbb{R}^2$, the weights and the control points are derived by the relations

$$(8) \qquad w_i = (P_i^w)_3, \qquad (P_i)_d = (P_i^w)/w_i, \quad d = 1, 2,$$

where w_i is called the ith weight and $(P_i)_d$ is the dth-dimension component of the vector P_i. The weight function $w(\xi)$ is defined as

$$(9) \qquad w(\xi) = \sum_{i=1}^{n} B_i^p(\xi) w_i.$$

Then, the NURBS basis functions and curve are defined by

$$(10) \qquad N_i^p(\xi) = \frac{B_i^p(\xi)w_i}{w(\xi)}, \qquad C(\xi) = \sum_{i=1}^n N_i^p(\xi)P_i.$$

The NURBS surfaces are analogously defined as follows

$$(11) \qquad S(\xi, \eta) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} N_{i,j}^{p_1,p_2}(\xi, \eta)P_{i,j},$$

where $N_{i,j}^{p_1,p_2}$ is the tensor product of NURBS basis functions $N_i^{p_1}$ and $N_j^{p_2}$. Similarly, NURBS solids can be defined by a three-dimensional tensor product. NURBS functions also satisfy the properties of B-spline functions. For a detailed exposition see, e.g. [35, 36, 37].

2.3. **Explicit Representation for B-splines.**

The recursive form of B-spline basis functions, given by (5), is elegant and concise, and is presented in all the IGA related references, see e.g., [15, 35, 36, 37]. However, this form may not be the most efficient from computational point of view, specially when dealing with large knot vectors. To the best of authors' knowledge, within the IGA literature there is no reference on the explicit representation of B-splines for a given mesh size h. Therefore, we present the explicit form of B-splines in terms of the mesh size (knot-span) h. Having an explicit form of basis functions is also advantageous in devising inter-grid transfer operators for multigrid and multilevel iterative solvers. For brevity reasons, we restrict ourselves to a unit interval with equal spacing. Moreover, as most of the NURBS based designs in engineering use polynomial degree $p = 2$ and 3, we will confine ourselves up to $p = 4$ with C^0 and C^{p-1} continuous basis functions.

2.3.1. C^{p-1}-*continuity.* We first consider the C^{p-1} continuous case as this is the default case for knot vector with non-repeated internal knots. For B-spline functions with $p = 0$ and $p = 1$, we have the same representation as for standard piecewise constant and linear finite element functions, respectively. Quadratic B-spline basis functions, however, differ from their FEA counterparts. They are each identical but shifted related to each other, whereas the shape of a quadratic finite element function depends on whether it corresponds to an internal node or an end node. This "homogeneous" pattern continues for the B-splines with higher-degrees.

We are interested to give an explicit representation for uniform B-spline basis functions defined on a knot vector E_k at any given level $l_k, k = 1, 2, 3, ...$, with spacing $h \ (= 1/n)$, where n is the total number of knot spans. We shall use the notation $B_{l,i}^{p,r}$ for B-splines, where superscripts represent the polynomial degree and the regularity of basis functions, respectively, and the subscripts represent the level and the number of basis function, respectively. We start with level l_1 with only one element. Using the definition from (5), at level l_1 the B-spline basis functions of degree $p = 2$ on the knot vector $E_1 = \{0, 0, 0, 1, 1, 1\}$ are defined as follows

$$(12) \qquad \begin{aligned} B_{l_1,1}^{2,p-1} &= (1-x)^2, \quad 0 \leq x \leq 1, \\ B_{l_1,2}^{2,p-1} &= 2x(1-x), \quad 0 \leq x \leq 1, \\ B_{l_1,3}^{2,p-1} &= x^2, \quad 0 \leq x \leq 1. \end{aligned}$$

The mesh refinement takes place by inserting the knots. We consider uniform refinement of E_1, i.e. inserting knots at the mid point of the knot values. At the next level l_2, the basis functions on refined

knot vector $E_2 = \{0, 0, 0, \frac{1}{2}, 1, 1, 1\}$ are given by

$$
B_{l_2,1}^{2,p-1} = \begin{cases} (1-2x)^2, & 0 \le x < \frac{1}{2}, \\ 0, & \frac{1}{2} \le x \le 1, \end{cases}
$$

$$
B_{l_2,2}^{2,p-1} = \begin{cases} 2x(2-3x), & 0 \le x < \frac{1}{2}, \\ 2(1-x)^2, & \frac{1}{2} \le x \le 1, \end{cases}
$$

(13)

$$
B_{l_2,3}^{2,p-1} = \begin{cases} 2x^2, & 0 \le x < \frac{1}{2}, \\ -2 + 8x - 6x^2, & \frac{1}{2} \le x \le 1, \end{cases}
$$

$$
B_{l_2,4}^{2,p-1} = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ (1-2x)^2, & \frac{1}{2} \le x \le 1. \end{cases}
$$

Further refinements take place in a similar way, i.e., starting with E_1, a single knot span, in the knot span E_k we will thus have 2^{k-1} knot spans. The explicit representation of B-splines at level $l_k, k \ge 3$, is given by

$$
B_{l_k,1}^{2,p-1} = \frac{1}{h^2}(h-x)^2, \quad 0 \le x < h,
$$

$$
B_{l_k,2}^{2,p-1} = \begin{cases} \dfrac{1}{2h^2}x(4h-3x), & 0 \le x < h, \\ \dfrac{1}{2h^2}(2h-x)^2, & h \le x < 2h, \end{cases} \quad \text{for } h \le \frac{1}{2},
$$

(14)

$$
B_{l_k,3+i}^{2,p-1} = \begin{cases} \dfrac{1}{2h^2}(x-ih)^2, & ih \le x < (i+1)h, \\ \dfrac{-3}{2} + \dfrac{3}{h}(x-ih) - \dfrac{1}{h^2}(x-ih)^2, & (i+1)h \le x < (i+2)h, \\ \dfrac{1}{2h^2}(3h-(x-ih))^2, & (i+2)h \le x < (i+3)h, \end{cases}
$$

$$
\text{where} \quad i = 0, 1, 2, 3, ..., (1/h) - 3, \text{ and } h \le 1/4.
$$

$$
B_{l_k,n+p-1}^{2,p-1} = \begin{cases} \dfrac{1}{2h^2}(-1+2h+x)^2, & 1-2h \le x < 1-h, \\ \dfrac{-1}{2h^2}(3-4h+2(2h-3)x+3x^2), & 1-h \le x \le 1, \end{cases} \quad \text{for } h \le \frac{1}{2},
$$

$$
B_{l_k,n+p}^{2,p-1} = \frac{1}{h^2}(1-h-x)^2, \quad 1-h \le x \le 1.
$$

For higher degree polynomials, we can define the explicit representation in a similar way. Again using the definition (5) of B-splines, for $p = 3$, at first level l_1 the basis functions with C^{p-1}-continuity are given as follows

$$
\begin{aligned}
B_{l_1,1}^{3,p-1} &= (1-x)^3, & 0 \le x \le 1, \\
B_{l_1,2}^{3,p-1} &= 3x(1-x)^2, & 0 \le x \le 1, \\
B_{l_1,3}^{3,p-1} &= 3x^2(1-x), & 0 \le x \le 1, \\
B_{l_1,4}^{3,p-1} &= x^3, & 0 \le x \le 1,
\end{aligned}
$$

(15)

and at level l_2, we have the following basis functions

$$B_{l_2,1}^{3,p-1} = \begin{cases} (1-2x)^3, & 0 \le x < \frac{1}{2}, \\ 0, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$B_{l_2,2}^{3,p-1} = \begin{cases} 2x(3 - 9x + 7x^2), & 0 \le x < \frac{1}{2}, \\ 2(1-x)^3, & \frac{1}{2} \le x \le 1, \end{cases}$$

(16)
$$B_{l_2,3}^{3,p-1} = \begin{cases} 2x^2(3 - 4x), & 0 \le x < \frac{1}{2}, \\ 2(-1+x)^2(-1+4x), & \frac{1}{2} \le x \le 1, \end{cases}$$

$$B_{l_2,4}^{3,p-1} = \begin{cases} 2x^3, & 0 \le x < \frac{1}{2}, \\ 2 - 12x + 24x^2 - 14x^3, & \frac{1}{2} \le x \le 1, \end{cases}$$

$$B_{l_2,5}^{3,p-1} = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ (-1+2x)^3, & \frac{1}{2} \le x \le 1. \end{cases}$$

For all other levels $l_k, k \ge 3$, the basis functions are defined below

(17)

$$B_{l_k,1}^{3,p-1} = \frac{1}{h^3}(h-x)^3, \quad 0 \le x < h,$$

$$B_{l_k,2}^{3,p-1} = \begin{cases} \dfrac{x}{h}\left(3 - \dfrac{9}{2}\dfrac{x}{h} + \dfrac{7}{4}\dfrac{x^2}{h^2}\right), & 0 \le x < h, \\ \dfrac{1}{4h^3}(-2h + x)^3, & h \le x < 2h, \end{cases} \quad \text{for } h \le \frac{1}{2},$$

$$B_{l_k,3}^{3,p-1} = \begin{cases} \dfrac{1}{6}\dfrac{x^2}{h^2}\left(9 - \dfrac{11}{2}\dfrac{x}{h}\right), & 0 \le x < h, \\ \dfrac{-3}{2} + \dfrac{9}{2}\dfrac{x}{h} - 3\dfrac{x^2}{h^2} + \dfrac{7}{4}\dfrac{x^3}{h^3}, & h \le x < 2h, \\ \dfrac{1}{6h^3}(-3h + x)^3, & 2h \le x < 3h, \end{cases} \quad \text{for } h \le \frac{1}{4},$$

$$B_{l_k,4+i}^{3,p-1} = \begin{cases} \dfrac{1}{6h^3}(x-ih)^3, & ih \le x < (i+1)h, \\ \dfrac{2}{3} - \dfrac{2}{h}(x-ih) + \dfrac{1}{2h^2}(x-ih)^2 - \dfrac{1}{2h^3}(x-ih)^3, & (i+1) \le x < (i+2)h, \\ \dfrac{-22}{3} + \dfrac{10}{h}(x-ih) - \dfrac{4}{h^2}(x-ih)^2 + \dfrac{1}{2h^3}(x-ih)^3, & (i+2)h \le x < (i+3)h, \\ \dfrac{32}{h}\left(1 - \dfrac{(x-ih)}{4h}\right)^3, & (i+3)h \le x < (i+4)h, \end{cases}$$

$$\text{where} \quad i = 0, 1, 2, 3, ..., (1/h) - 4, \text{ and } h \le \frac{1}{4},$$

$$B_{l_k,n+p-2}^{3,p-1} = \begin{cases} \dfrac{1}{6h^3}(-3h + (1-x))^3, & 1 - 3h \le x < 1 - 2h, \\ \dfrac{-3}{2} + \dfrac{9}{2}\dfrac{(1-x)}{h} - 3\dfrac{(1-x)^2}{h^2} + \dfrac{7}{4}\dfrac{(1-x)^3}{h^3}, & 1 - 2h \le x < 1 - h, \\ \dfrac{1}{6}\dfrac{(1-x)^2}{h^2}\left(9 - \dfrac{11}{2}\dfrac{(1-x)}{h}\right), & 1 - h \le x \le 1, \end{cases}$$

$$\text{for } h \le \frac{1}{4},$$

$$B_{l_k,n+p-1}^{3,p-1} = \begin{cases} \dfrac{1}{4h^3}(-2h + (1-x))^3, & 1 - 2h \le x < 1 - h, \\ \dfrac{(1-x)}{h}\left(3 - \dfrac{9}{2}\dfrac{(1-x)}{h} + \dfrac{7}{4}\dfrac{(1-x)^2}{h^2}\right), & 1 - h \le x < 1, \end{cases} \quad \text{for } h \le \frac{1}{2},$$

$$B_{l_k,n+p}^{3,p-1} = \frac{1}{h^3}(h - (1-x))^3, \quad 1 - h \le x \le 1.$$

Finally, we give the explicit representation of basis functions for $p = 4$ with C^{p-1}-continuity. At level l_1, with knot E_1, the B-spline basis functions of degree $p = 4$ are given by

(18)
$$
\begin{aligned}
B_{l_1,1}^{4,p-1} &= (1-x)^4, \quad 0 \le x \le 1, \\
B_{l_1,2}^{4,p-1} &= 4x(1-x)^3, \quad 0 \le x \le 1, \\
B_{l_1,3}^{4,p-1} &= 6x^2(1-x)^2, \quad 0 \le x \le 1, \\
B_{l_1,4}^{4,p-1} &= 4x^3(1-x), \quad 0 \le x \le 1, \\
B_{l_1,5}^{4,p-1} &= x^4, \quad 0 \le x \le 1.
\end{aligned}
$$

The B-splines on second level l_2 with knot E_2 are defined as follows

(19)
$$
\begin{aligned}
B_{l_2,1}^{4,p-1} &= \begin{cases} (1-2x)^4, & 0 \le x < \frac{1}{2}, \\ 0, & \frac{1}{2} \le x \le 1, \end{cases} \\[2mm]
B_{l_2,2}^{4,p-1} &= \begin{cases} 2x(4 - 18x + 28x^2 - 15x^3), & 0 \le x < \frac{1}{2}, \\ 2(1-x)^4, & \frac{1}{2} \le x \le 1, \end{cases} \\[2mm]
B_{l_2,3}^{4,p-1} &= \begin{cases} 2x^2(6 - 16x + 11x^2), & 0 \le x < \frac{1}{2}, \\ 2(1-x)^3(-1+5x), & \frac{1}{2} \le x \le 1, \end{cases} \\[2mm]
B_{l_2,4}^{4,p-1} &= \begin{cases} 2x^3(4-5x), & 0 \le x < \frac{1}{2}, \\ 2(1-x)^2(1 - 6x + 11x^2), & \frac{1}{2} \le x \le 1, \end{cases} \\[2mm]
B_{l_2,5}^{4,p-1} &= \begin{cases} 2x^4, & 0 \le x < \frac{1}{2}, \\ -2 + 16x - 48x^2 + 64x^3 - 30x^4, & \frac{1}{2} \le x \le 1, \end{cases} \\[2mm]
B_{l_2,6}^{4,p-1} &= \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ (1-2x)^4, & \frac{1}{2} \le x \le 1. \end{cases}
\end{aligned}
$$

At all other levels $l_k, k \ge 3$, the basis functions of degree $p = 4$ with C^{p-1}-continuity are given by

$$
B_{l_k,1}^{4,p-1} = \frac{1}{h^4}(h-x)^4, \quad 0 \le x < h,
$$

$$
B_{l_k,2}^{4,p-1} = \begin{cases} \dfrac{-4x}{h}\left(-1 + \dfrac{9}{4}\dfrac{x}{h} - \dfrac{7}{4}\dfrac{x^2}{h^2} + \dfrac{15}{32}\dfrac{x^3}{h^3}\right), & 0 \le x < h, \\[3mm] \dfrac{1}{8h^4}(2h-x)^4, & h \le x < 2h, \end{cases} \quad \text{for } h \le \frac{1}{2},
$$

$$
B_{l_k,3}^{4,p-1} = \begin{cases} \dfrac{1}{9}\dfrac{x^2}{h^2}\left(27 - 33\dfrac{x}{h} + \dfrac{85}{8}\dfrac{x^2}{h^2}\right), & 0 \le x < h, \\[3mm] \dfrac{-3}{2} + 6\dfrac{x}{h} - 6\dfrac{x^2}{h^2} + \dfrac{7}{3}\dfrac{x^3}{h^3} - \dfrac{23}{72}\dfrac{x^4}{h^4}, & h \le x < 2h, \\[3mm] \dfrac{1}{18h^4}(3h-x)^4, & 2h \le x < 3h, \end{cases} \quad \text{for } h \le \frac{1}{4},
$$

$$
B_{l_k,4}^{4,p-1} = \begin{cases} \dfrac{2}{3}\dfrac{x^3}{h^3} - \dfrac{25}{72}\dfrac{x^4}{h^4}, & 0 \le x < h, \\[3mm] \dfrac{2}{3} - \dfrac{8}{3}\dfrac{x}{h} + 4\dfrac{x^2}{h^2} - 2\dfrac{x^3}{h^3} + \dfrac{23}{72}\dfrac{x^4}{h^4}, & h \le x < 2h, \\[3mm] \dfrac{-22}{3} + \dfrac{40}{3}\dfrac{x}{h} - 8\dfrac{x^2}{h^2} + 2\dfrac{x^3}{h^3} - \dfrac{13}{72}\dfrac{x^4}{h^4}, & 2h \le x < 3h, \\[3mm] \dfrac{1}{24h^4}(4h-x)^4, & 3h \le x < 4h, \end{cases} \quad \text{for } h \le \frac{1}{4}.
$$

$$B_{l_k,5+i}^{4,p-1} = \begin{cases} \dfrac{1}{24h^4}(x-ih)^4, & ih \leq x < (i+1)h, \\[2mm] \dfrac{1}{24}\left(-5 + \dfrac{20}{h}(x-ih) - \dfrac{30}{h^2}(x-ih)^2 + \dfrac{20}{h^3}(x-ih)^3 - \dfrac{4}{h^4}(x-ih)^4\right), \\[1mm] & (i+1)h \leq x < (i+2)h, \\[2mm] \dfrac{155}{24} - \dfrac{25}{2h}(x-ih) + \dfrac{35}{4h^2}(x-ih)^2 - \dfrac{5}{2h^3}(x-ih)^3 - \dfrac{1}{4h^4}(x-ih)^4, \\[1mm] & (i+2)h \leq x < (i+3)h, \\[2mm] \dfrac{-655}{24} + \dfrac{65}{2h}(x-ih) - \dfrac{55}{4h^2}(x-ih)^2 + \dfrac{5}{2h^3}(x-ih)^3 - \dfrac{1}{6h^4}(x-ih)^4, \\[1mm] & (i+3)h \leq x < (i+4)h, \\[2mm] \dfrac{1}{24h^4}(5h-(x-ih))^4, & (i+4)h \leq x < (i+5)h, \end{cases}$$

$$\text{where} \quad i = 0,1,2,3,...,(1/h)-5, \text{ and } h \leq \dfrac{1}{4}.$$

(20)

$$B_{l_k,n+p-3}^{4,p-1} = \begin{cases} \dfrac{1}{24h^4}\left(4h-(1-x)\right)^4, & 1-4h \leq x < 1-3h, \\[2mm] \dfrac{-22}{3} + \dfrac{40}{3}\dfrac{(1-x)}{h} - 8\dfrac{(1-x)^2}{h^2} + 2\dfrac{(1-x)^3}{h^3} - \dfrac{13}{72}\dfrac{(1-x)^4}{h^4}, \\[1mm] & 1-3h \leq x < 1-2h, \\[2mm] \dfrac{2}{3} - \dfrac{8}{3}\dfrac{(1-x)}{h} + 4\dfrac{(1-x)^2}{h^2} - 2\dfrac{(1-x)^3}{h^3} + \dfrac{23}{72}\dfrac{(1-x)^4}{h^4}, \\[1mm] & 1-2h \leq x < 1-h, \\[2mm] \dfrac{2}{3}\dfrac{(1-x)^3}{h^3} - \dfrac{25}{72}\dfrac{(1-x)^4}{h^4}, & 1-h \leq x \leq 1, \end{cases}$$

$$\text{for } h \leq \dfrac{1}{4},$$

$$B_{l_k,n+p-2}^{4,p-1} = \begin{cases} \dfrac{1}{18h^4}(3h-(1-x))^4, & 1-3h \leq x < 1-2h, \\[2mm] \dfrac{-3}{2} + 6\dfrac{(1-x)}{h} - 6\dfrac{(1-x)^2}{h^2} + \dfrac{7}{3}\dfrac{(1-x)^3}{h^3} - \dfrac{23}{72}\dfrac{(1-x)^4}{h^4}, \\[1mm] & 1-2h \leq x < 1-h, \\[2mm] \dfrac{1}{9}\dfrac{(1-x)^2}{h^2}\left(27 - 33\dfrac{(1-x)}{h} + \dfrac{85}{8}\dfrac{(1-x)^2}{h^2}\right), & 1-h \leq x \leq 1, \end{cases}$$

$$\text{for } h \leq \dfrac{1}{4},$$

$$B_{l_k,n+p-1}^{4,p-1} = \begin{cases} \dfrac{1}{8h^4}(2h-(1-x))^4, & 1-2h \leq x < 1-h, \\[2mm] \dfrac{-4(1-x)}{h}\left(-1 + \dfrac{9}{4}\dfrac{(1-x)}{h} - \dfrac{7}{4}\dfrac{(1-x)^2}{h^2} + \dfrac{15}{32}\dfrac{(1-x)^3}{h^3}\right), \\[1mm] & 1-h \leq x \leq 1, \end{cases}$$

$$\text{for } h \leq \dfrac{1}{2},$$

$$B_{l_k,n+p}^{4,p-1} = \dfrac{1}{h^4}(h-(1-x))^4, \quad 0 \leq x \leq 1.$$

2.3.2. C^0-*continuity*. To lower the continuity of the basis functions across element boundaries, the knot values are repeated upto a desired level. By repeating the internal knots r times we get the C^{p-r} continuous basis functions. In the previous section we have given the explicit representation for C^{p-1} continuous B-splines, which is the highest continuity for polynomial degree p. We now consider another extreme case, the lowest continuity, i.e. C^0 continuous basis functions. At first level l_1 the C^0 continuous B-spline basis functions of degree $p = 2,3,4$ on a knot $E_1 = \{0,0,0,1,1,1\}$ are same as those of C^{p-1} continuous B-spline basis functions of same degree.

The explicit representation for C^0 continuous B-spline basis functions of degree $p = 2$ at level $l_k, k \geq 2$ is given by

$$B^{2,0}_{l_k,1} = \frac{1}{h^2}(h - x)^2, \quad 0 \leq x < h,$$

$$B^{2,0}_{l_k,2+2i} = \frac{-2}{h^2}(x - ih)(h + (x - ih)), \quad (i-1)h \leq x < ih,$$

$$\text{where } i = 0, 1, 2, 3, ..., (1/h) - 1,$$

(21)

$$B^{2,0}_{l_k,3+2i} = \begin{cases} \frac{1}{h^2}(h + (x - ih))^2, & (i-1)h \leq x < ih, \\ \frac{1}{h^2}(-h + (x - ih))^2, & ih \leq x < (i+1)h, \end{cases}$$

$$\text{where } i = 0, 1, 2, 3, ..., ((1/h) - 2),$$

$$B^{2,0}_{l_k,np+1} = \frac{1}{h^2}(1 - h - x)^2, \quad 1 - h \leq x \leq 1.$$

For $p = 3$, the explicit representation for B-spline basis functions with C^0-continuity, at level $l_k, k \geq 2$, are given by

$$B^{3,0}_{l_k,1} = \frac{1}{h^3}(h - x)^3, \quad 0 \leq x < h,$$

$$B^{3,0}_{l_k,2+3i} = \frac{3}{h}\left(-1 + \frac{1}{h}(x - ih)\right)^2, \quad ih \leq x < (i+1)h,$$

$$\text{where } i = 0, 1, 2, 3, ..., (1/h) - 1,$$

$$B^{3,0}_{l_k,3+3i} = \frac{3}{h^2}(x - ih)^2\left(1 - \frac{1}{h}(x - ih)\right), \quad ih \leq x < (i+1)h,$$

$$\text{where } i = 0, 1, 2, 3, ..., (1/h) - 1,$$

(22)

$$B^{3,0}_{l_k,4+3i} = \begin{cases} \frac{1}{h^3}(x - ih)^3, & ih \leq x < (i+1)h, \\ 8\left(1 - \frac{1}{2h}(x - ih)\right)^3, & (i+1)h \leq x < (i+2)h, \end{cases}$$

$$\text{where } i = 0, 1, 2, 3, ..., ((1/h) - 2),$$

$$B^{3,0}_{l_k,np+1} = \frac{1}{h^3}(1 - h - x)^3, \quad 1 - h \leq x \leq 1.$$

Finally, the explicit representation for C^0 continuous basis functions of degree $p = 4$ at level $l_k, k \geq 2$, is given below

$$B^{4,0}_{l_k,1} = \frac{1}{h^4}(h - x)^4, \quad 0 \leq x < h,$$

$$B^{4,0}_{l_k,2+4i} = \frac{4}{h}(x - ih)\left(1 - \frac{(x - ih)}{h}\right)^3, \quad ih \leq x < (i+1)h,$$

$$\text{where } i = 0, 1, 2, 3, ..., (1/h) - 1,$$

$$B^{4,0}_{l_k,3+4i} = \frac{6}{h^2}(x - ih)^2\left(1 - \frac{(x - ih)}{h}\right)^2, \quad ih \leq x < (i+1)h,$$

$$\text{where } i = 0, 1, 2, 3, ..., (1/h) - 1,$$

$$B^{4,0}_{l_k,4+4i} = \frac{4}{h^3}(x - ih)^3\left(1 - \frac{(x - ih)}{h}\right), \quad ih \leq x < (i+1)h,$$

$$\text{where } i = 0, 1, 2, 3, ..., (1/h) - 1,$$

$$(23) \quad B^{4,0}_{l_k,5+4i} = \begin{cases} \dfrac{1}{h^4}(x - ih)^4, & ih \le x < (i+1)h, \\[2mm] 16\left(1 - \dfrac{1}{2h}(x - ih)\right)^4, & (i+1)h \le x < (i+2)h, \end{cases}$$
$$\text{where } i = 0, 1, 2, 3, ..., ((1/h) - 2),$$

$$B^{4,0}_{l_k,np+1} = \frac{1}{h^4}(1 - h - x)^4, \quad 1 - h \le x \le 1.$$

3. Multilevel Representation of B-splines and NURBS

3.1. **Multilevel B-splines.** In this section, we study the multilevel structure of B-splines and NURBS spaces. This will be used in the construction of corresponding hierarchical spaces (i.e. splitting the fine space into coarse space and its hierarchical complement) in Section 5. For a two level setting, let $\mathcal{B}^{p,r}_c$ and $\mathcal{B}^{p,r}_f$ denote the B-spline spaces at coarse and fine level, respectively. Let $\{B^{p,r}_{c,i}, i = 1, 2, ..., n_c\}$ and $\{B^{p,r}_{f,i}, i = 1, 2, ..., n_f\}$ be the set of basis functions for coarse and fine space, respectively, i.e.

$$\mathcal{B}^{p,r}_c = \text{span}\{B^{p,r}_{c,1}, B^{p,r}_{c,2}, B^{p,r}_{c,3}, ..., B^{p,r}_{c,n_c}\},$$

and

$$\mathcal{B}^{p,r}_f = \text{span}\{B^{p,r}_{f,1}, B^{p,r}_{f,2}, B^{p,r}_{f,3}, ..., B^{p,r}_{f,n_f}\}.$$

The following result expresses coarse basis functions as the linear combination of fine basis functions.

Proposition 3. *Each coarse basis function $B^{p,r}_{c,i}, i = 1, 2, ..., n_c$, can be represented as the linear combination of the fine basis functions $\{B^{p,r}_{f,i}, i = 1, 2, ..., n_f\}$ by the following relation*

$$(24) \quad \mathcal{B}^{p,r}_c = G^{p,r}_f \mathcal{B}^{p,r}_f, \quad \text{i.e.,} \quad B^{p,r}_{c,i} = \sum_{j=1}^{n_f} g_{ij} B^{p,r}_{f,j},$$

where $G^{p,r}_f = (g_{ij})_{n_c \times n_f}$, is called the restriction operator from a given fine level to the next coarse level for B-spline basis functions.

In the following we explain the formation of transfer operator $G^{p,r}_f$ at different levels of mesh and with increasing polynomial degree with both the extreme cases of C^{p-1} and C^0-continuity.

3.1.1. C^{p-1}-*continuity.* The B-spline basis functions $B^{2,p-1}_{l_1,i}, i = 1, 2, 3$, and $B^{2,p-1}_{l_2,i}, i = 1, 2, 3, 4$, of degree $p = 2$ on knots $E_1 = \{0, 0, 0, 1, 1, 1\}$ and $E_2 = \{0, 0, 0, \frac{1}{2}, 1, 1, 1\}$, respectively, are defined in section 2.3.1. Clearly, the total number of coarse and fine basis functions are three ($n_c = 3$) and four ($n_f = 4$), respectively. The matrix $G^{2,p-1}_{l_2} = (g_{ij})_{3 \times 4}$ is given by the following representation of coarse basis functions as the linear combination of fine basis functions.

$$\boxed{\begin{aligned} B^{2,p-1}_{l_1,1} &= g_{11}B^{2,p-1}_{l_2,1} + g_{12}B^{2,p-1}_{l_2,2} + g_{13}B^{2,p-1}_{l_2,3} + g_{14}B^{2,p-1}_{l_2,4}, \\ B^{2,p-1}_{l_1,1} &= g_{21}B^{2,p-1}_{l_2,1} + g_{22}B^{2,p-1}_{l_2,2} + g_{23}B^{2,p-1}_{l_2,3} + g_{24}B^{2,p-1}_{l_2,4}, \\ B^{2,p-1}_{l_1,1} &= g_{31}B^{2,p-1}_{l_2,1} + g_{32}B^{2,p-1}_{l_2,2} + g_{33}B^{2,p-1}_{l_2,3} + g_{34}B^{2,p-1}_{l_2,4}. \end{aligned}}$$

Equivalently, it can be written as

$$\mathcal{B}^{2,p-1}_{l_1} = G^{2,p-1}_{l_2} \mathcal{B}^{2,p-1}_{l_2},$$

where

$$\mathcal{B}^{2,p-1}_{l_1} = \begin{bmatrix} B^{2,p-1}_{l_1,1} \\ B^{2,p-1}_{l_1,2} \\ B^{2,p-1}_{l_1,3} \end{bmatrix}, G^{2,p-1}_{l_2} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \end{bmatrix}, \mathcal{B}^{2,p-1}_{l_2} = \begin{bmatrix} B^{2,p-1}_{l_2,1} \\ B^{2,p-1}_{l_2,2} \\ B^{2,p-1}_{l_2,3} \\ B^{2,p-1}_{l_2,4} \end{bmatrix}.$$

For the above set of basis functions, $G^{2,p-1}_{l_2}$ is given by

$$(25a) \quad G^{2,p-1}_{l_2} = \left(\frac{1}{4}\right) \times \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

Similarly, the coarse basis functions for $B_{l_2,i}^{2,p-1}, i = 1, 2, 3, 4$, at level l_2, can be obtained in terms of $B_{l_3,i}^{2,p-1}, i = 1, 2, ..., 6$, by the following matrix

$$(25b) \qquad G_{l_3}^{2,p-1} = \left(\frac{1}{4}\right) \times \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}.$$

In a multilevel setting, the representation of each basis function $B_{l_k,i}^{2,p-1}$ at level $l_k, k \geq 3$, as the linear combination of the basis functions $B_{l_{k+1},i}^{2,p-1}$ at level l_{k+1} is given by the the following matrix $G_{l_{k+1}}^{2,p-1}$.

$$(25c) \qquad G_{l_{k+1}}^{2,p-1} = \left(\frac{1}{4}\right) \times \begin{bmatrix} 4 & 2 & & & & & & & & \\ & 2 & 3 & 1 & & & & & & \\ & & 1 & 3 & 3 & 1 & & & & \\ & & & 1 & 3 & 3 & 1 & & & \\ & & & & .. & .. & .. & .. & & \\ & & & & & .. & .. & .. & .. & \\ & & & & & & 1 & 3 & 3 & 1 \\ & & & & & & & 1 & 3 & 3 & 1 \\ & & & & & & & & 1 & 3 & 2 \\ & & & & & & & & & 2 & 4 \end{bmatrix}.$$

The size of the matrix $G_{l_{k+1}}^{2,p-1}$ is $(n_{l_k} + 2) \times (n_{l_{k+1}} + 2)$, where n_{l_k} and $n_{l_{k+1}}$ are the number of total knot spans at level l_k and l_{k+1}, respectively.

For higher degree polynomials, the transfer operators can be defined in a similar way. For $p = 3$, at level l_1 the basis functions $B_{l_1,i}^{3,p-1}, i = 1, 2, 3, 4$, with C^{p-1}-continuity can be represented by the following restriction operator

$$(26a) \qquad G_{l_2}^{3,p-1} = \left(\frac{1}{2}\right) \times \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

The transfer operator for level l_3 can be written as

$$(26b) \qquad G_{l_3}^{3,p-1} = \left(\frac{1}{16}\right) \times \begin{bmatrix} 16 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 12 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 10 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 12 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 16 \end{bmatrix}.$$

For all levels l_{k+1} with $k \geq 3$, we have

$$(26c) \qquad G_{l_{k+1}}^{3,p-1} = \left(\frac{1}{16}\right) \times \begin{bmatrix} 16 & 8 & & & & & & & & \\ & 8 & 12 & 3 & & & & & & \\ & & 4 & 11 & 8 & 2 & & & & \\ & & & 2 & 8 & 12 & 8 & 2 & & \\ & & & & .. & .. & .. & .. & .. & \\ & & & & & .. & .. & .. & .. & .. \\ & & & & & & 2 & 8 & 12 & 8 & 2 \\ & & & & & & & 2 & 8 & 11 & 4 \\ & & & & & & & & 3 & 12 & 8 \\ & & & & & & & & & 8 & 16 \end{bmatrix}.$$

The size of the matrix $G_{l_{k+1}}^{3,p-1}$ is $(n_{l_k} + 3) \times (n_{l_{k+1}} + 3)$.

Finally, we give the transfer operators for $p = 4$ with C^{p-1}-continuity. For levels l_2 and l_3 the transfer operators are defined as follows:

$$
(27a) \qquad G_{l_2}^{4,p-1} = \left(\frac{1}{2}\right) \times
\begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix},
$$

$$
(27b) \qquad G_{l_3}^{4,p-1} = \left(\frac{1}{48}\right) \times
\begin{bmatrix}
48 & 24 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24 & 36 & 9 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 30 & 9 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 30 & 12 & 0 & 0 \\
0 & 0 & 0 & 0 & 9 & 36 & 24 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 24 & 48
\end{bmatrix},
$$

respectively. For levels $l_k, k \geq 3$, the transfer operator is given by the following
(27c)

$$
G_{l_{k+1}}^{4,p-1} = \left(\frac{1}{48}\right) \times
\begin{bmatrix}
48 & 24 & & & & & & & & & \\
24 & 36 & 9 & & & & & & & & \\
12 & 33 & 20 & 4 & & & & & & & \\
6 & 25 & 29 & 15 & 3 & & & & & & \\
3 & 15 & 30 & 30 & 15 & 3 & & & & & \\
& .. & .. & .. & .. & & & & & & \\
& .. & .. & .. & .. & & & & & & \\
& & 3 & 15 & 30 & 30 & 15 & 3 & & & \\
& & & 3 & 15 & 29 & 25 & 6 & & & \\
& & & & 4 & 20 & 33 & 12 & & & \\
& & & & & 9 & 36 & 24 & & & \\
& & & & & & 24 & 48 & & &
\end{bmatrix},
$$

where the size of the matrix is $(n_{l_k} + 4) \times (n_{l_{k+1}} + 4)$.

Remark 4. *Since in the span of an internal basis function of degree p at coarse level, there are $p + 2$ full basis functions in the same span at fine level, therefore, any row of $G_{l_{k+1}}^{p,p-1}$ can have at most $p + 2$ entries.*

3.1.2. C^0*-continuity.* In section 2.3.2, we explained the explicit representation of C^0 continuous B-spline basis functions. The corresponding transfer operators are given in this section. The transfer operator $G_{l_2}^{2,0}$ for $p = 2$ with C^0-continuity at level l_2 is given by

$$
(28a) \qquad G_{l_2}^{2,0} = \left(\frac{1}{4}\right) \times
\begin{bmatrix}
4 & 2 & 1 & 0 & 0 \\
0 & 2 & 2 & 2 & 0 \\
0 & 0 & 1 & 2 & 4
\end{bmatrix}.
$$

The operator $G_{l_{k+1}}^{2,0}$ for $k \geq 2$, in general, is given by

$$
(28b) \qquad G_{l_{k+1}}^{2,0} = \left(\frac{1}{4}\right) \times
\begin{bmatrix}
4 & 2 & 1 & 0 & 0 & & & & & \\
0 & 2 & 2 & 2 & 0 & & & & & \\
0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & \\
& & & & 0 & 2 & 2 & 2 & 0 & \\
& & & & 0 & 0 & 1 & 2 & 4 & \\
& & & & .. & .. & .. & & & \\
& & & & .. & .. & .. & & & \\
& & & & & & 4 & 2 & 1 & 0 & 0 \\
& & & & & & 0 & 2 & 2 & 2 & 0 \\
& & & & & & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 \\
& & & & & & & & & & 0 & 2 & 2 & 2 & 0 \\
& & & & & & & & & & 0 & 0 & 1 & 2 & 4
\end{bmatrix},
$$

with size $(2n_{l_k} + 1) \times (2n_{l_{k+1}} + 1)$. The matrix $G_{l_{k+1}}^{2,0}, k \geq 2$, has block structure with blocks $G_{l_2}^{2,0}$. The blocks are connected in such a way that if a block ends at ith row and jth column of $G_{l_{k+1}}^{2,0}$ then the next block will start at (i,j)th position of $G_{l_{k+1}}^{2,0}$ with an overlap of last entry and first entry of the corresponding blocks. Note that, the first entry and the last entry in a block are same.

The transfer operators for $p = 3$ with C^0-continuity for levels l_2 and l_3 are given by

$$
(29a) \qquad G_{l_2}^{3,0} = \left(\frac{1}{8}\right) \times \begin{bmatrix} 8 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & 8 \end{bmatrix},
$$

and

$$
(29b) \qquad G_{l_3}^{3,0} = \left(\frac{1}{8}\right) \times \begin{bmatrix} \boxed{\begin{matrix} 8 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & \end{matrix}} \begin{matrix} \\ \\ \\ \boxed{\begin{matrix} 8 & 4 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & 8 \end{matrix}} \end{matrix} \end{bmatrix},
$$

respectively. Following the same block structure as in $G_{l_{k+1}}^{2,0}$, we can generate $G_{l_{k+1}}^{3,0}$ with size $(3n_{l_k} + 1) \times (3n_{l_{k+1}} + 1)$. Finally for $p = 4$, we have the following transfer operators for levels l_2 and l_3

$$
(30a) \qquad G_{l_2}^{4,0} = \left(\frac{1}{16}\right) \times \begin{bmatrix} 16 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 8 & 6 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 6 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 16 \end{bmatrix},
$$

and

$$
(30b) \qquad G_{l_3}^{4,0} = \left(\frac{1}{16}\right) \times \begin{bmatrix} \boxed{\begin{matrix} 16 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 8 & 6 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 6 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & \end{matrix}} \\ \boxed{\begin{matrix} 16 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 8 & 6 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 6 & 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 16 \end{matrix}} \end{bmatrix},
$$

respectively. Similarly, repeating these blocks as in previous cases, we can generate $G_{l_{k+1}}^{4,0}$ with size $(4n_{l_k} + 1) \times (4n_{l_{k+1}} + 1)$.

Remark 5. *Note that the transfer operators are defined for one dimensional B-splines. For two- and three-dimensions, we take tensor product of these operators.*

3.2. **Multilevel NURBS.** This section presents the procedure for constructing NURBS multilevel spaces in a simplified manner. Since NURBS are generated from B-splines, its natural to construct NURBS transfer operators from B-splines transfer operators. For a two level setting, let $\mathcal{N}_c^{p,r}$ and $\mathcal{N}_f^{p,r}$ denote the NURBS spaces at coarse and fine level, respectively. Let $\{N_{c,i}^{p,r}, i = 1, 2, ..., n_c\}$ and $\{N_{f,i}^{p,r}, i = 1, 2, ..., n_f\}$ be the set of basis functions for coarse and fine space, respectively, i.e.

$$
\mathcal{N}_c^{p,r} = \text{span}\{N_{c,1}^{p,r}, N_{c,2}^{p,r}, N_{c,3}^{p,r}, ..., N_{c,n_c}^{p,r}\},
$$

and

$$
\mathcal{N}_f^{p,r} = \text{span}\{N_{f,1}^{p,r}, N_{f,2}^{p,r}, N_{f,3}^{p,r}, ..., N_{f,n_f}^{p,r}\}.
$$

Note that, Proposition 3 holds for NURBS basis functions also, i.e., we have

$$(31) \qquad \mathcal{N}_c^{p,r} = R_f^{p,r} \mathcal{N}_f^{p,r}, \quad \text{i.e.,} \quad N_{c,i}^{p,r} = \sum_{j=1}^{n_f} r_{ij} N_{f,j}^{p,r}, \quad \forall i = 1, 2, 3, ..., n_c,$$

where $R_f^{p,r} = (r_{ij})_{n_c \times n_f}$, is restriction operator with respect to NURBS basis functions. As NURBS are formed from B-splines and 'weights', $R_f^{p,r}$ can be obtained from $G_f^{p,r}$ and 'weights'. Using the definition of NURBS and (31), we have

$$(32) \qquad \frac{w_i^c B_{c,i}^{p,r}}{\displaystyle\sum_{i'=1}^{n_c} w_{i'}^c B_{c,i'}^{p,r}} = \sum_{j=1}^{n_f} r_{ij} \frac{w_j^f B_{f,j}^{p,r}}{\displaystyle\sum_{j'=1}^{n_f} w_{j'}^f B_{f,j'}^{p,r}}, \quad \forall i = 1, 2, 3, ..., n_c,$$

where $w_i^c, i = 1, 2, 3, ..., n_c$, and $w_j^f, j = 1, 2, 3, ..., n_f$, are the weights for coarse space and fine space, respectively. Note that the weight function $\sum_{i=1}^{n} w_i B_i$ does not change its value with respect to refinements, i.e., we have

$$(33) \qquad \sum_{i=1}^{n_c} w_i^c B_{c,i}^{p,r} = \sum_{j=1}^{n_f} w_j^f B_{f,j}^{p,r},$$

which is an important result from the refinement point of view. Now using (33), from (32) we get

$$w_i^c B_{c,i}^{p,r} = \sum_{j=1}^{n_f} r_{ij} w_j^f B_{f,j}^{p,r},$$

and thus

$$(34) \qquad B_{c,i}^{p,r} = \sum_{j=1}^{n_f} \frac{r_{ij} w_j^f}{w_i^c} B_{f,j}^{p,r}.$$

Comparing the coefficients of $B_{f,j}^{p,r}$ in (24) and (34), we get

$$(35) \qquad \frac{r_{ij} w_j^f}{w_i^c} = g_{ij} \implies r_{ij} = \frac{g_{ij} w_i^c}{w_j^f}.$$

This can be equivalently written as

$$(36) \qquad R_f^{p,r} = W_I^c G_f^{p,r} \left(W_I^f \right)^{-1},$$

where W_I^c and W_I^f are the diagonal matrices corresponding to weights at the coarse level and fine level, respectively, and defined as follows

$$W_I^c = \begin{bmatrix} w_1^c & & & & & \\ & w_2^c & & & & \\ & & .. & & & \\ & & & .. & & \\ & & & & w_{n_c-1}^c & \\ & & & & & w_{n_c}^c \end{bmatrix}, W_I^f = \begin{bmatrix} w_1^f & & & & & \\ & w_2^f & & & & \\ & & .. & & & \\ & & & .. & & \\ & & & & w_{n_f-1}^f & \\ & & & & & w_{n_f}^f \end{bmatrix}.$$

The equation (36) gives us the relation between B-splines and NURBS operators using B-splines transfer operators and weights at coarse and fine levels. From (33) we can also obtain the procedure to refine the weights as follows. We have

$$\sum_{i=1}^{n_c} w_i^c B_{c,i}^{p,r} = \sum_{j=1}^{n_f} w_j^f B_{f,j}^{p,r},$$

14

which implies

$$\sum_{i=1}^{n_c} w_i^c \sum_{j=1}^{n_f} g_{ij} B_{f,j}^{p,r} = \sum_{j=1}^{n_f} w_j^f B_{f,j}^{p,r}.$$

Comparing the coefficients of $B_{f,j}^{p,r}$ from both the sides, we get

$$(37) \qquad w_j^f = \sum_{i=1}^{n_c} w_i^c g_{ij} \quad \text{for } j = 1, 2, ..., n_f.$$

Equivalently, the above can be written in matrix form as follows

$$(38) \qquad W^f = \left(G_f^{p,r} \right)^T W^c,$$

where

$$W^f = \begin{bmatrix} w_1^f \\ w_2^f \\ \vdots \\ \vdots \\ w_{n_f-1}^f \\ w_{n_f}^f \end{bmatrix}, W^c = \begin{bmatrix} w_1^c \\ w_2^c \\ \vdots \\ \vdots \\ w_{n_c-1}^c \\ w_{n_c}^c \end{bmatrix}.$$

Using above, now we can write the NURBS operators in terms of B-spline operator and weights only at coarse level. From (35), we get

$$(39) \qquad r_{ij} = \frac{g_{ij} w_i^c}{\displaystyle\sum_{i=1}^{n_c} w_i^c g_{ij}}.$$

In matrix form the above can be written as

$$(40) \qquad R_f^{p,r} = W_I^c G_f^{p,r} \left(\text{diag} \left(\left(G_f^{p,r} \right)^T W^c \right) \right)^{-1}.$$

Remark 6. *The operators $G_f^{p,r}$ and $R_f^{p,r}$ can also be used in constructing restriction operators in multi-grid methods, see e.g.,* [24].

4. AMLI METHODS

In this section we present the basic principle of AMLI methods. In what follows we will denote by $M^{(l_k)}$ a preconditioner for stiffness matrix $A^{(l_k)}$ corresponding to level l_k. We will also make use of the corresponding hierarchical matrix $\hat{A}^{(l_k)}$, which is related to $A^{(l_k)}$ via a two-level hierarchical basis (HB) transformation $J^{(l_k)}$, i.e.,

$$(41) \qquad \hat{A}^{(l_k)} = J^{(l_k)} A^{(l_k)} (J^{(l_k)})^T.$$

The transformation matrix $J^{(l_k)}$ specifies the space splitting, which will be described in detail in Section 5. By $A_{ij}^{(l_k)}$ and $\hat{A}_{ij}^{(l_k)}$, $1 \le i, j \le 2$, we denote the blocks of $A^{(l_k)}$ and $\hat{A}^{(l_k)}$ that correspond to the fine-coarse partitioning of degrees of freedom (DOF) where the DOF associated with the coarse mesh are numbered last.

The aim is to build a multilevel preconditioner $M^{(f)}$ for the coefficient matrix $A^{(f)} := A_h$ at the level of the finest mesh that has a uniformly bounded (relative) condition number

$$\varkappa(M^{(f)^{-1}} A^{(f)}) = \mathcal{O}(1),$$

and an optimal computational complexity, that is, linear in the number of degrees of freedom N_f at the finest mesh (grid). In order to achieve this goal hierarchical basis methods can be combined with various types of stabilization techniques.

One particular purely algebraic stabilization technique is the so-called algebraic multilevel iteration (AMLI) method, where a specially constructed matrix polynomial $p^{(l_k)}$ of degree ν_k can be employed at some (or all) levels l_k. The AMLI algorithm has been originally introduced and studied in a multiplicative form, see [3, 4].

We have the following two-level hierarchical basis representation at level l_k

$$(42) \qquad \hat{A}^{(l_k)} = \begin{bmatrix} \hat{A}_{11}^{(l_k)} & \hat{A}_{12}^{(l_k)} \\ \hat{A}_{21}^{(l_k)} & \hat{A}_{22}^{(l_k)} \end{bmatrix} = \begin{bmatrix} A_{11}^{(l_k)} & \hat{A}_{12}^{(l_k)} \\ \hat{A}_{21}^{(l_k)} & A^{(l_{k-1})} \end{bmatrix}.$$

Starting at level l_1 (associated with the coarsest mesh), on which a complete LU factorization of the matrix $A^{(l_1)}$ is performed, we define

$$(43) \qquad M^{(l_1)} := A^{(l_1)}.$$

Given the preconditioner $M^{(l_{k-1})}$ at level l_{k-1}, the preconditioner $M^{(l_k)}$ at level l_k is then defined by

$$(44) \qquad M^{(l_k)} := L^{(l_k)} U^{(l_k)},$$

where

$$(45) \qquad L^{(l_k)} := \begin{bmatrix} C_{11}^{(l_k)} & 0 \\ \hat{A}_{21}^{(l_k)} & C_{22}^{(l_k)} \end{bmatrix}, \quad U^{(l_k)} := \begin{bmatrix} I & C_{11}^{(l_k)^{-1}} \hat{A}_{12}^{(l_k)} \\ 0 & I \end{bmatrix}.$$

Here $C_{11}^{(l_k)}$ is a preconditioner for the pivot block $A_{11}^{(l_k)}$, and

$$(46) \qquad C_{22}^{(l_k)} := A^{(l_{k-1})} \left(I - p^{(l_k)} (M^{(l_{k-1})^{-1}} A^{(l_{k-1})}) \right)^{-1}$$

$$(47) \qquad 0 \le p^{(l_k)}(t) < 1, \quad 0 < t \le 1, \quad p^{(l_k)}(0) = 1.$$

It is easily seen that (46) is equivalent to

$$(48) \qquad C_{22}^{(l_k)^{-1}} = M^{(l_{k-1})^{-1}} q^{(l_k)} (A^{(l_{k-1})} M^{(l_{k-1})^{-1}}),$$

where the polynomial $q^{(l_k)}$ is given by

$$(49) \qquad q^{(l_k)}(x) = \frac{1 - p^{(l_k)}(x)}{x}.$$

We note that the multilevel preconditioner defined via (44) is getting close to a two-level method when $q^{(l_k)}(x)$ closely approximates $1/x$, in which case $C_{22}^{(l_k)^{-1}} \approx A^{(l_{k-1})^{-1}}$. In order to construct an efficient multilevel method, the action of $C_{22}^{(l_k)^{-1}}$ on an arbitrary vector should be much cheaper to compute (in terms of the number of arithmetic operations) than the action of $A^{(l_{k-1})^{-1}}$. Optimal order solution algorithms typically require that the arithmetic work for one application of $C_{22}^{(l_k)^{-1}}$ is of the order $\mathcal{O}(N_{l_{k-1}})$ where $N_{l_{k-1}}$ denotes the number of unknowns at level l_{k-1}.

It is well known from the theory introduced in [3, 4] that a properly shifted and scaled Chebyshev polynomial $p^{(l_k)} := p_{\nu_k}$ of degree ν_k can be used to stabilize the condition number of $M^{(l_k)^{-1}} \hat{A}^{(l_k)}$ (and thus obtain optimal order computational complexity). Other polynomials such as the best polynomial approximation of $1/x$ in uniform norm also qualify for stabilization, see, e.g., [32]. Alternatively, in the nonlinear AMLI method, see, e.g., [6], a few inner flexible conjugate gradient (FCG) type iterations (for the FCG algorithm, see also [33]) are performed in order to improve (or freeze) the residual reduction factor of the outer FCG iteration. In general, the resulting nonlinear (variable step) multilevel preconditioning method is almost equally efficient, and, because its realization does not rely on any spectral bounds, is easier to implement than the linear AMLI method (based on a stabilization polynomial). For a convergence analysis of nonlinear AMLI see, e.g., [30, 38].

The construction of optimal preconditioners in the framework of AMLI methods is based upon a theory in which the constant γ in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality plays a key role. The CBS constant measures the cosine of the abstract angle between the coarse space and its hierarchical complementary space. The general idea is to construct a proper splitting by means of a hierarchical basis transformation.

In the hierarchical bases context we denote by V_1 and V_2 subspaces of the space V_h. The space V_2 is spanned by the coarse-space basis functions and V_1 is the complement of V_2 in V_h, i.e., V_h is a direct sum of V_1 and V_2:

$$V_h = V_1 \oplus V_2.$$

Let $v_i \in V_i, i = 1, 2$. The CBS constant measures the strength of the off-diagonal blocks in relation to the diagonal blocks (see, (42)) and can be defined as the minimal γ satisfying the strengthened CBS inequality

$$(50) \qquad |v_1^T \hat{A}_{12} v_2| \leq \gamma \left\{ (v_1^T \hat{A}_{11} v_1)(v_2^T \hat{A}_{22} v_2) \right\}^{1/2}$$

A detailed exposition about the role of this constant can be found in [22].

Typically, the iterative solution process is of optimal order of computational complexity if the degree $\nu_k = \nu$ of the matrix polynomial (or alternatively, the number of inner iterations for nonlinear AMLI) at level l_k satisfies the optimality condition

$$(51) \qquad 1/\sqrt{(1 - \gamma^2)} < \nu < \tau,$$

where $\tau \approx \tau_k = N_{l_k}/N_{l_{k-1}}$ denotes the reduction factor of the number of degrees of freedom (DOF), and γ denotes the constant in the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality. The value of τ is approximately 4 and 8 in case of two- and three-dimensional problems, respectively. For a more detailed discussion of AMLI methods, including implementation issues, see, e.g., [31, 38].

Remark 7. *The preconditioner defined in* (44) *is of multiplicative form. The introduction of AMLI methods was based on the multiplicative form, see* [3, 4, 5, 6], *and is commonly used in practice. However, it is also possible to choose the preconditioner in the additive form, which is defined as follows*

$$(52) \qquad M_A^{(l_k)} := \begin{bmatrix} C_{11}^{(l_k)} & 0 \\ 0 & C_{22}^{(l_k)} \end{bmatrix}.$$

In this case the optimal order of computational complexity demands that the matrix polynomial degree (or the number of inner iterations of nonlinear AMLI) satisfy the following relation

$$(53) \qquad \sqrt{(1 + \gamma)/(1 - \gamma)} < \nu < \tau.$$

5. CONSTRUCTION OF HIERARCHICAL SPACES

The hierarchical basis techniques result in splittings in which the angle between the coarse space and its hierarchical complement is uniformly bounded with respect to the mesh size. Recall from section 4, we have the following two-level hierarchical basis representation for stiffness matrix at fine level

$$\hat{A}^{(f)} = \begin{bmatrix} \hat{A}_{11}^{(f)} & \hat{A}_{12}^{(f)} \\ \hat{A}_{21}^{(f)} & \hat{A}_{22}^{(f)} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}^{(f)} & \hat{A}_{12}^{(f)} \\ \hat{A}_{21}^{(f)} & A^{(c)} \end{bmatrix},$$

where $\hat{A}_{22}^{(f)}$ represents the matrix corresponding to coarse basis functions and $\hat{A}_{11}^{(f)}$ represents the matrix corresponding to its hierarchical complement. Recall from Section 3, for B-splines and NURBS we have the following transformations

$$(54) \qquad \begin{aligned} \hat{A}_{22}^{(f)} = A^{(c)} = G_f^{p,r} A^f (G_f^{p,r})^T, \\ \hat{A}_{22}^{(f)} = A^{(c)} = R_f^{p,r} A^f (R_f^{p,r})^T, \end{aligned}$$

respectively. For hierarchical complementary spaces, let $T_f^{p,r}$ be the matrix such that

$$(55) \qquad \hat{A}_{11}^{(f)} = T_f^{p,r} A^f (T_f^{p,r})^T.$$

Here the matrix $T_f^{p,r}$ is a hierarchical complementary transfer operator, which transfers fine basis functions to a set of hierarchical complementary basis functions. The remaining two blocks of the hierarchical matrix $\hat{A}^{(f)}$ can be obtained by the following relations

$$(56) \qquad \begin{aligned} \hat{A}_{12}^{(f)} = T_f^{p,r} A^{(f)} (G_f^{p,r})^T, \\ \hat{A}_{21}^{(f)} = G_f^{p,r} A^{(f)} (T_f^{p,r})^T. \end{aligned}$$

To construct $T_f^{p,r}$ efficiently, the following points are important.

(1) The condition number of the $\hat{A}_{11}^{(f)}$ block should be independent of mesh size.

(2) The CBS constant γ, see (50), should be bounded away from one, i.e. the minimum generalized eigenvalue of block $\hat{A}_{22}^{(f)}$ with respect to the Schur complement should be greater than $1/4$ for $\nu = 2$ and $1/9$ for $\nu = 3$.

(3) The basis for hierarchical complementary space should be locally supported. In other words, the block $\hat{A}_{11}$ should be sparse and nicely structured.

The construction of $T_f^{p,r}$, based on the linear combination of fine basis functions, is not unique. Based on the above mentioned guidelines, a representation of a complementary basis function should not involve several fine basis functions, because it will cause less sparse structure of matrix $T_f^{p,r}$. Based on our extensive study with different choices of linear combinations satisfying the above requirements, we present two choices for $T_f^{p,r}$, for $p = 2, 3, 4$ and for extreme cases of smoothness, namely C^{p-1} and C^0.

For the first choice, we have the following matrix representation of hierarchical complementary space for $p = 2$ with C^{p-1}-continuity.

$$T_{l_{k+1}}^{2,p-1} = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{matrix}} & & & \\ & & .. & .. & & \\ & & .. & .. & & \\ & & & & \boxed{\begin{matrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{matrix}} \end{bmatrix}.$$

The above matrix has the block structure with blocks, say $M_1^{2,p-1}$. The blocks are connected in such a way that if a block ends at ith row and jth column of $T_{l_{k+1}}^{2,p-1}$ then the next block will start at $(i+1, j-1)$th position of $T_{l_{k+1}}^{2,p-1}$. In general, for $p = 2, 3, 4$, we write the following block form of $T_{l_{k+1}}^{p,p-1}$ with blocks $M_1^{p,p-1}$

$$(57) \qquad T_{l_{k+1}}^{p,p-1} = \begin{bmatrix} M_1^{p,p-1} & & & & \\ & M_1^{p,p-1} & & & \\ & & .. & & \\ & & & M_1^{p,p-1} & \\ & & & & M_1^{p,p-1} \end{bmatrix},$$

where

$$M_1^{2,p-1} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

$$M_1^{3,p-1} = \begin{bmatrix} 0 & -1/2 & 3/4 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 3/4 & -1/2 & 0 \end{bmatrix},$$

and

$$M_1^{4,p-1} = \begin{bmatrix} 0 & 1/2 & -1 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & -1 & 1 & -1/2 & 0 \end{bmatrix},$$

respectively. The blocks are connected in such a way that if a block ends at ith row and jth column of $T_{l_{k+1}}^{p,p-1}$ then the next block will start at $(i + 1, j - (p - 1))$th position of $T_{l_{k+1}}^{p,p-1}$.

For second choice of $T_f^{p,r}$ we give the following block matrix

$$(58) \qquad T_{l_{k+1}}^{p,p-1} = \begin{bmatrix} M_2^{p,p-1} & & & & \\ & M_2^{p,p-1} & & & \\ & & .. & & \\ & & & M_2^{p,p-1} & \\ & & & & M_2^{p,p-1} \end{bmatrix},$$

where the blocks $M_2^{p,p-1}$ are given by

$$M_2^{2,p-1} = \begin{bmatrix} -1/2 & 1 & -1 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -1 & 1/2 \end{bmatrix},$$

$$M_2^{3,p-1} = \begin{bmatrix} 1/8 & -1/2 & 3/4 & -1/2 & 1/8 & 0 & 0 \\ 0 & 0 & 1/8 & -1/2 & 3/4 & -1/2 & 1/8 \end{bmatrix},$$

and

$$M_2^{4,p-1} = \begin{bmatrix} 1/4 & 1/2 & -1 & 1 & -1/2 & -1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 1/2 & -1 & 1 & -1/2 & -1/4 \end{bmatrix},$$

respectively.

For C^0 continuous basis functions, we give the following matrix representation of hierarchical complementary spaces. For $p = 2$ we have

$$T_{l_{k+1}}^{2,0} = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & -1/4 & 0 & 0 \\ 0 & 0 & 1 & -1/4 & 0 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} 0 & 1 & -1/4 & 0 & 0 \\ 0 & 0 & 1 & -1/4 & 0 \end{matrix}} & & \\ & & \ddots & \ddots & \\ & & & \boxed{\begin{matrix} 0 & 1 & -1/4 & 0 & 0 \\ 0 & 0 & 1 & -1/4 & 0 \end{matrix}} \end{bmatrix}.$$

The above matrix has the block structure and the blocks are connected in such a way that if a block ends at ith row and jth column of $T_{l_{k+1}}^{2,0}$ then the next block will start at $(i+1, j)$th position of $T_{l_{k+1}}^{2,0}$. In general, for $p = 2, 3, 4$, we can write the following hierarchical complementary operators

$$(59) \qquad T_{l_{k+1}}^{p,0} = \begin{bmatrix} M_1^{p,0} & & & & \\ & M_1^{p,0} & & & \\ & & \ddots & & \\ & & & M_1^{p,0} & \\ & & & & M_1^{p,0} \end{bmatrix},$$

where

$$M_1^{2,0} = \begin{bmatrix} 0 & 1 & -1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/4 & 1 & 0 \end{bmatrix},$$

$$M_1^{3,0} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

and

$$M_1^{4,0} = \begin{bmatrix} 0 & -2/3 & 5/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & 5/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/4 & -2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5/4 & -2/3 & 0 \end{bmatrix},$$

respectively. Another choice for $T_f^{p,r}$ for C^0 continuous basis functions is obtained by choosing the following block matrix

$$(60) \qquad T_{l_{k+1}}^{p,0} = \begin{bmatrix} M_2^{p,0} & & & & \\ & M_2^{p,0} & & & \\ & & \ddots & & \\ & & & M_2^{p,0} & \\ & & & & M_2^{p,0} \end{bmatrix},$$

where

$$M_2^{2,0} = \begin{bmatrix} -1/4 & 1 & -1/4 & 0 & 0 \\ 0 & 0 & -1/4 & 1 & -1/4 \end{bmatrix},$$

$$M_2^{3,0} = \begin{bmatrix} 0 & -1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/4 & 1/10 & -1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 & 0 \end{bmatrix},$$

19

and

$$M_2^{4,0} = \begin{bmatrix} 0 & -5/9 & 1 & -5/9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5/9 & 1 & -5/9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5/9 & 1 & -5/9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5/9 & 1 & -5/9 & 0 \end{bmatrix},$$

respectively.

Remark 8. *All the above operators are defined for one-dimension. The higher dimensional operators are obtained via tensor product.*

6. NUMERICAL RESULTS

To test the performance of the AMLI methods, we consider the following test problems, whose discretizations are performed using the Matlab toolbox GeoPDEs [21, 23].

Example 9. *Let $\Omega = (0,1)^2$. Together with $\mathcal{A} = I$, and Dirichlet boundary conditions, the right hand side function f is chosen such that the analytical solution of the problem is given by $u = e^x \sin(y)$.*

Example 10. *The domain is chosen as a quarter annulus in the first Cartesian quadrant with inner radius 1 and outer radius 2. Together with $\mathcal{A} = I$, and homogeneous Dirichlet boundary conditions, the right hand side function f is chosen such that the analytic solution is given by $u = -xy^2(x^2 + y^2 - 1)(x^2 + y^2 - 4)$, see [21, 23].*

Example 11. *The domain is chosen as a thick quarter of a ring. Together with $\mathcal{A} = I$, and Dirichlet boundary conditions, the right hand side function f is chosen such that the analytical solution of the problem is given by $u = e^x \sin(xy)cos(z)$.*

At the finest level (largest problem size), the parametric domain is divided into n equal elements in each direction. The initial guess for (iteratively) solving the linear system of equations is chosen as the zero vector. Let r_0 denote the initial residual vector and r_{it} denote the residual vector at a given PCG/FCG iteration n_{it}. The following stopping criteria is used

(61)
$$\frac{\|r_{\mathrm{it}}\|}{\|r_0\|} \leq 10^{-8}.$$

The average convergence factor reported in the following tables is defined as $\rho = \left(\frac{\|r_{\mathrm{it}}\|}{\|r_0\|} \right)^{1/n_{\mathrm{it}}}$. In the following tables, by L1, L2 and N2 we denote the linear multiplicative AMLI cycles with $\nu = 1$, $\nu = 2$ and non-linear multiplicative AMLI cycle with $\nu = 2$, respectively. By t_c, we represent the setup time, i.e., the time taken in the construction of transfer operators and generating the preconditioner for $\hat{A}_{11}$ block (for which we used the ILU(0) factorization, i.e. without any fill-in). The solver time is represented by t_s. For all the test cases we take the polynomial degree $p = 2, 3, 4$ with C^0- and C^{p-1}-continuity. Furthermore, the transfer operator $G_{l_{k+1}}^{p,r}$ is fixed and it exactly represents the coarse basis functions in the space of fine basis functions. The hierarchical complementary transfer operator $T_{l_{k+1}}^{p,r}$ are chosen in two different ways as defined in Section 5, see (57)-(60).

We first consider the Example 1 and provide t_c, t_s, n_{it} and ρ for L1-, L2-, N2- cycles with both the choices of $T_{l_{k+1}}^{p,r}$. Numerical results are presented in Tables 1-2 and Tables 3-4 for first choice and second choice of $T_{l_{k+1}}^{p,r}$, respectively . From Tables 1-4 we observe the following:

- The number of iterations and total solution $(t_c + t_s)$ time show an h-independent convergence rates for C^{p-1}- and C^0-continuity.
- For C^{p-1}-continuity, the results are almost p-independent, whereas for C^0-continuity, the degree p has some effect on PCG/FCG iterations.
- For C^{p-1}-continuity, all the AMLI cycles give optimal results, and the V-cycle ($\nu = 1$) is the fastest among all. This is due to a very nice bound on γ for C^{p-1}-continuity. Therefore, in the remaining numerical computations we consider linear AMLI cycle with $\nu = 1$ and nonlinear AMLI cycle with $\nu = 2$ for C^{p-1} continuous basis functions.
- For C^0-continuity, V-cycle ($\nu = 1$) is not an optimal order method, an observation similar to standard FEM. However, for C^0-continuity, both the $\nu = 2$ cycle methods (linear and nonlinear) exhibit optimal order behavior, and nonlinear AMLI gives overall better results. Therefore, we

TABLE 1. AMLI methods for Example 1: First choice of $T_{l_{k+1}}^{p,r}$ given in (57) with C^{p-1} regularity

$1/h$	t_c	t_s			n_{it}			ρ		
		L1	L2	N2	L1	L2	N2	L1	L2	N2
$p = 2$										
8	0.00	0.00	0.00	0.00	7	7	7	0.0641	0.0641	0.0622
16	0.00	0.00	0.01	0.01	8	7	7	0.0948	0.0966	0.0670
32	0.01	0.01	0.01	0.01	9	8	7	0.1108	0.0988	0.0672
64	0.04	0.02	0.03	0.04	9	8	7	0.1086	0.0901	0.0622
128	0.18	0.07	0.09	0.12	9	8	7	0.1166	0.0909	0.0624
256	0.72	0.25	0.30	0.41	9	8	7	0.1175	0.0879	0.0603
512	2.97	1.03	1.12	1.50	9	8	7	0.1276	0.0945	0.0620
$p = 3$										
8	0.00	0.00	0.00	0.00	8	8	8	0.0901	0.0901	0.0901
16	0.01	0.01	0.01	0.01	9	9	8	0.1111	0.1129	0.0686
32	0.02	0.01	0.01	0.02	10	9	7	0.1293	0.1043	0.0577
64	0.10	0.03	0.04	0.05	10	8	7	0.1361	0.0857	0.0551
128	0.41	0.12	0.13	0.18	10	8	7	0.1369	0.0821	0.0536
256	1.76	0.48	0.46	0.63	10	8	7	0.1348	0.0794	0.0523
512	7.50	1.65	1.77	2.37	9	8	7	0.1283	0.0771	0.0511
$p = 4$										
8	0.00	0.00	0.00	0.00	10	10	10	0.1139	0.1139	0.1139
16	0.01	0.01	0.01	0.01	12	12	10	0.1866	0.1882	0.1378
32	0.06	0.02	0.02	0.03	12	11	9	0.2013	0.1822	0.1100
64	0.26	0.07	0.07	0.10	12	10	9	0.2038	0.1557	0.1032
128	1.09	0.26	0.24	0.37	12	9	9	0.2028	0.1209	0.0977
256	4.57	0.98	0.88	1.21	12	9	8	0.1976	0.1182	0.0975
512	19.05	3.60	3.44	4.66	11	9	8	0.1853	0.1146	0.0930

consider only nonlinear AMLI cycle with $\nu = 2$ for C^0 continuous basis functions in remaining numerical results.

- For $p = 4$ with C^{p-1}-continuity, we could not obtain better γ with the second choice of $T_{l_{k+1}}^{p,r}$ as compared to the first choice. Therefore, in Table 3, the numerical results are presented only for $p = 2, 3$ with second choice of $T_{l_{k+1}}^{p,r}$. Numerical results for $p = 4$ may be improved by choosing different operators, which demands further investigation.

- For C^{p-1}-continuity, though the number of iterations are less for second choice of $T_{l_{k+1}}^{p,r}$, the overall time $(t_c + t_s)$ is more than the first choice of $T_{l_{k+1}}^{p,r}$. This happens due to comparatively less sparse structure of second choice $T_{l_{k+1}}^{p,r}$, which results in more construction time t_c. Therefore, in the remaining numerical tests we consider only the first choice of $T_{l_{k+1}}^{p,r}$ for C^{p-1} continuous basis functions.

- For C^0-continuity, we get mixed results from both the choices of $T_{l_{k+1}}^{p,r}$. This is due to the fact that there is not much difference in number of nonzero entries in each row of $T_{l_{k+1}}^{p,r}$ for two different choices. Therefore, numerical results are provided for both the choices of $T_{l_{k+1}}^{p,r}$ for C^0 continuous basis functions.

We now consider Example 2 with curved boundary. The geometry for this example is represented by NURBS basis functions of order 1 in the radial direction and of order 2 in the angular direction, see [21]. Numerical results are provided for C^{p-1}-continuity with first choice of $T_{l_{k+1}}^{p,r}$ in Table 5, and for C^0-continuity with both the choices of $T_{l_{k+1}}^{p,r}$ in Table 6. All the results are qualitatively similar to that of Example 1 with square domain.

Finally, we consider three-dimensional problem as stated in Example 3. The numerical results are presented in Tables 7-8. Due to the limitation of computer resources numerical results in three-dimensions

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*did not converge upto desired accuracy.

TABLE 2. AMLI methods for Example 1: First choice of $T_{l_{k+1}}^{p,r}$ given in (59) with C^0 regularity

$1/h$	t_c	t_s			n_{it}			ρ		
		L1	L2	N2	L1	L2	N2	L1	L2	N2
					$p=2$					
8	0.00	0.01	0.01	0.01	9	9	9	0.1072	0.1072	0.1072
16	0.01	0.01	0.01	0.01	11	11	9	0.1695	0.1716	0.1102
32	0.02	0.03	0.04	0.04	13	11	9	02195	0.1738	0.1110
64	0.10	0.07	0.09	0.10	14	11	9	0.2606	0.1744	0.1109
128	0.38	0.30	0.29	0.34	16	11	9	0.2973	0.1743	0.1105
256	1.65	1.25	1.04	1.23	17	11	9	0.3288	0.1736	0.1102
512	6.93	5.17	3.84	4.61	18	11	9	0.3557	0.1730	0.1100
					$p=3$					
8	0.01	0.01	0.01	0.03	12	12	12	0.1999	0.1999	0.1999
16	0.02	0.03	0.04	0.03	17	17	12	0.3288	0.3305	0.2124
32	0.09	0.09	0.12	0.10	22	18	12	0.4258	0.3568	0.2129
64	0.37	0.38	0.41	0.33	27	19	12	0.5014	0.3650	0.2122
128	1.55	1.77	1.34	1.21	32	19	12	0.5581	0.3673	0.2114
256	6.73	7.87	4.88	4.51	37	19	12	0.6038	0.3670	0.2110
512	28.76	36.56	19.12	17.84	42	19	12	0.6394	0.3664	0.2108
					$p=4$					
8	0.01	0.03	0.03	0.03	19	19	19	0.3631	0.3631	0.3631
16	0.05	0.07	0.10	0.09	25	26	19	0.4784	0.4827	0.3719
32	0.24	0.32	0.36	0.29	38	30	19	0.6087	0.5337	0.3720
64	1.07	1.62	1.29	1.05	52	32	19	0.6982	0.5585	0.3719
128	4.50	8.04	4.90	3.89	67	34	19	0.7585	0.5766	0.3709
256	18.73	40.11	19.41	15.25	85	35	19	0.8038	0.5827	0.3703
512	76.22	190.29	77.37	62.24	100*	35	19	0.8379	0.5878	0.3700

TABLE 3. AMLI methods for Example 1: Second choice of $T_{l_{k+1}}^{p,r}$ given in (58) with C^{p-1} regularity

$1/h$	t_c	t_s			n_{it}			ρ		
		L1	L2	N2	L1	L2	N2	L1	L2	N2
					$p=2$					
8	0.08	0.02	0.42	0.52	5	5	5	0.0227	0.0227	0.0227
16	0.00	0.01	0.01	0.01	6	6	5	0.0304	0.0326	0.0217
32	0.02	0.01	0.01	0.01	6	6	5	0.0316	0.0311	0.0226
64	0.07	0.02	0.05	0.05	6	6	5	0.0303	0.0300	0.0224
128	0.30	0.06	0.08	0.10	6	6	5	0.0314	0.0310	0.0234
256	1.21	0.22	0.30	0.39	6	6	5	0.0301	0.0296	0.0226
512	5.18	0.88	1.05	1.62	6	6	6	0.0326	0.0321	0.0269
					$p=3$					
8	0.00	0.02	0.00	0.00	7	7	7	0.0443	0.0443	0.0443
16	0.01	0.00	0.00	0.01	7	7	6	0.0560	0.0569	0.0365
32	0.04	0.01	0.01	0.02	7	7	6	0.0576	0.0494	0.0319
64	0.18	0.04	0.04	0.05	7	6	5	0.0569	0.0377	0.0216
128	0.77	0.12	0.13	0.17	7	6	5	0.0542	0.0343	0.0204
256	3.32	0.47	0.49	0.64	7	6	5	0.0502	0.0326	0.0195
512	13.99	1.60	1.89	2.44	6	6	5	0.0446	0.0311	0.0186

are provided only upto $h = 1/32$. In Table 7, linear AMLI cycle with $\nu = 1$, and nonlinear AMLI cycle with $\nu = 2$ are given for C^{p-1} continuity with first choice of $T_{l_{k+1}}^{p,r}$. The results exhibit optimal order for

TABLE 4. AMLI methods for Example 1: Second choice of $T^{p,r}_{l_{k+1}}$ given in (60) with C^0 regularity

$1/h$	t_c	t_s			n_{it}			ρ		
		L1	L2	N2	L1	L2	N2	L1	L2	N2
					$p=2$					
8	0.00	0.00	0.00	0.00	8	8	8	0.0901	0.0901	0.0901
16	0.01	0.01	0.01	0.01	9	9	8	0.1173	0.1195	0.0918
32	0.04	0.02	0.03	0.04	10	9	8	0.1308	0.1197	0.0900
64	0.15	0.07	0.09	0.12	10	9	8	0.1430	0.1192	0.0890
128	0.62	0.27	0.32	0.41	10	9	8	0.1479	0.1191	0.0884
256	2.71	1.08	1.20	1.56	10	9	8	0.1509	0.1191	0.0880
512	11.22	4.33	4.55	6.00	10	9	8	0.1528	0.1191	0.0878
					$p=3$					
8	0.01	0.01	0.03	0.03	9	9	9	0.1133	0.1133	0.1133
16	0.02	0.02	0.02	0.02	11	11	9	0.1724	0.1743	0.1191
32	0.09	0.05	0.07	0.07	13	11	9	0.2216	0.1762	0.1206
64	0.39	0.20	0.22	0.25	14	11	9	0.2627	0.1777	0.1212
128	1.62	0.88	0.79	0.91	16	11	9	0.2998	0.1782	0.1215
256	7.06	3.70	2.87	3.42	17	11	9	0.3321	0.1785	0.1216
512	29.78	16.54	11.24	13.37	19	11	9	0.3630	0.1786	0.1217
					$p=4$					
8	0.01	0.02	0.02	0.02	10	10	10	0.1368	0.1368	0.1368
16	0.07	0.04	0.06	0.05	13	13	10	0.2199	0.2219	0.1419
32	0.33	0.15	0.18	0.18	15	13	10	0.2825	0.2254	0.1416
64	1.88	0.61	0.60	0.64	17	13	10	0.3259	0.2251	0.1412
128	5.92	2.51	2.20	2.41	18	13	10	0.3575	0.2247	0.1410
256	25.51	11.39	8.56	9.56	20	13	10	0.3844	0.2245	0.1408
512	104.33	49.49	35.15	39.67	21	13	10	0.4042	0.2244	0.1408

both the solvers. The increased number of iterations (as compared to two-dimensional examples) can be attributed to the smaller angle between coarse space and its complementary space. For C^0-continuity the numerical results with both the choices of $T^{p,r}_{l_{k+1}}$ are given in Table 8. The first choice of $T^{p,r}_{l_{k+1}}$, however, does not result in an optimal order method. The optimality is restored with $\nu = 3$, which are presented in the column with $N3$. The second choice, though expensive, gives optimal order method for second order stabilization ($\nu = 2$). In Tables 7-8, The entries marked by $*$ represent the cases where the computations are performed on a machine with larger memory but shared with other users, therefore timings are not provided for these cases.

We note that for two-dimensional problems, the total time of the solvers also exhibit optimal complexity, however, for three-dimensional problem the increase in the total time ($t_c + t_s$) for successive refinement is more than the factor of increase in number of unknowns. This is due to the construction of operators $G^{p,r}_{l_{k+1}}$ and $T^{p,r}_{l_{k+1}}$ by tensor product of matrices for one-dimensional operators (see Remark 8), and expensive preconditioner for $\hat{A}_{11}$ (ILU(0)). In our future study on local analysis, we also intend to construct these operators for two- and three-dimensional problems without tensor product, and devise efficient and cheaper preconditioner for $\hat{A}_{11}$.

7. CONCLUSIONS

We have presented AMLI methods for the linear system arising from the isogeometric discretization of the scalar second order elliptic problems. We summarize the main contribution of this paper as follows.

(1) We provide the explicit representation of B-splines as a function of mesh size h on a unit interval with uniform refinement. The explicit representation is given for C^0 and C^{p-1} continuous basis functions of polynomial degree $p = 2, 3, 4$, the most widely used cases in engineering applications. Explicit form of B-splines is important from computational point of view, as well as in

TABLE 5. AMLI methods for Example 2: First choice of $T_{l_{k+1}}^{p,r}$ given in (57) with C^{p-1} regularity

$1/h$	t_c	t_s		n_{it}		ρ	
		L1	N2	L1	N2	L1	N2
$p = 2$							
8	0.02	0.02	0.01	8	8	0.0802	0.0802
16	0.00	0.01	0.01	9	8	0.1201	0.0839
32	0.01	0.01	0.01	10	7	0.1499	0.0658
64	0.05	0.02	0.03	11	6	0.1838	0.0453
128	0.17	0.09	0.10	12	6	0.2048	0.0351
256	0.72	0.38	0.30	13	5	0.2211	0.0226
512	2.93	1.53	1.07	13	5	0.2374	0.0194
$p = 3$							
8	0.00	0.00	0.00	9	9	0.1201	0.1201
16	0.01	0.01	0.01	10	9	0.1560	0.1148
32	0.02	0.01	0.02	12	8	0.1839	0.0988
64	0.10	0.04	0.06	13	8	0.2104	0.0900
128	0.41	0.16	0.20	13	8	0.2363	0.0858
256	1.76	0.66	0.72	14	8	0.2514	0.0828
512	7.45	2.56	2.35	14	7	0.2644	0.0706
$p = 4$							
8	0.03	0.01	0.00	11	11	0.1686	0.1686
16	0.01	0.01	0.01	12	11	0.2073	0.1665
32	0.05	0.02	0.03	13	9	0.2419	0.1248
64	0.24	0.11	0.10	14	9	0.2549	0.1054
128	1.07	0.32	0.43	15	8	0.2688	0.0884
256	4.47	1.23	1.09	15	7	0.2924	0.0648
512	18.79	5.30	4.13	16	7	0.3061	0.0534

forming the inter-grid transfer operators. It is intended to help the reader in writing optimized/fast computer programs.

(2) The construction of B-spline basis functions at coarse level from the linear combination of fine basis functions is provided. For $p = 2, 3, 4$, and with C^0 and C^{p-1} continuities, these transfer operators (from fine level to coarse level) are given in matrix form for a multilevel mesh. These operators can also be used to generate restriction operators in multigrid methods.

(3) The transfer operators are also provided for NURBS basis functions. The formulation of NURBS operators is given in terms of B-spline operators and weights.

(4) The construction of hierarchical spaces for B-splines (NURBS) is presented. Hierarchical spaces are constructed as direct sum of coarse spaces and corresponding hierarchical complementary spaces. We have presented matrix form of these operators. As the choice of hierarchical complementary spaces is not unique, we have provided two different choices of these operators for each of C^0- and C^{p-1}-continuity of basis functions.

(5) For a given polynomial degree p, AMLI cycles are of optimal complexity with respect to the mesh refinement. Also, for a given mesh size h, AMLI cycles are (almost) p-independent. We provided numerical results for a square domain, quarter annulus (ring), and quarter thick ring. The iteration counts, convergence factor, and timings are given for AMLI linear V-, W- and nonlinear W-cycles. Note that, for C^{p-1}-continuity the linear V-cycle also exhibits optimal convergence rates (due to very nice space splitting, which is normally not found in standard FEM). The linear and nonlinear AMLI W-cycle is optimal for all cases except for a particular case of degree $p = 4$ with C^0-continuity in three-dimensional problem with first choice of $T_f^{p,r}$. For this case, the number of iterations are provided with $\nu = 3$ cycle, which is optimal. The numerical results are complete for $p = 2, 3, 4$, with C^{p-1} and C^0 continuous basis functions.

Despite that the condition number of the stiffness matrix grows very rapidly with the polynomial degree, these excellent results exhibit the strength and flexibility of AMLI methods. Nevertheless, the

TABLE 6. AMLI methods for Example 2: C^0 regularity

with first choice of $T_{l_{k+1}}^{p,r}$ given in (59)					with second choice of $T_{l_{k+1}}^{p,r}$ given in (60)				
$1/h$	t_c	t_s	n_{it}	ρ	$1/h$	t_c	t_s	n_{it}	ρ
		N2	N2	N2			N2	N2	N2
$p=2$					$p=2$				
8	0.00	0.01	11	0.1744	8	0.00	0.01	10	0.1445
16	0.01	0.02	11	0.1820	16	0.01	0.02	10	0.1510
32	0.02	0.05	11	0.1791	32	0.04	0.04	10	0.1478
64	0.09	0.13	11	0.1752	64	0.15	0.15	10	0.1463
128	0.40	0.43	11	0.1730	128	0.62	0.52	10	0.1437
256	1.72	1.52	11	0.1717	256	2.65	1.93	10	0.1419
512	7.36	5.61	11	0.1704	512	11.05	7.72	10	0.1401
$p=3$					$p=3$				
8	0.00	0.01	13	0.2237	8	0.01	0.01	11	0.1647
16	0.02	0.04	14	0.2507	16	0.02	0.03	11	0.1780
32	0.08	0.11	14	0.2584	32	0.09	0.09	11	0.1845
64	0.34	0.39	14	0.2632	64	0.39	0.33	12	0.1883
128	1.49	1.43	14	0.2649	128	1.63	1.21	12	0.1922
256	6.35	5.37	14	0.2648	256	6.98	4.52	12	0.1938
512	27.51	20.83	14	0.2638	512	28.76	17.94	12	0.1940
$p=4$					$p=4$				
8	0.01	0.03	22	0.4319	8	0.01	0.02	11	0.1660
16	0.05	0.11	24	0.4516	16	0.07	0.05	11	0.1758
32	0.22	0.38	24	0.4563	32	0.32	0.19	11	0.1789
64	0.92	1.34	24	0.4591	64	1.39	0.70	11	0.1785
128	4.21	5.03	24	0.4609	128	5.99	2.64	11	0.1774
256	18.28	19.79	24	0.4639	256	25.31	10.49	11	0.1765
512	76.62	81.78	25	0.4644	512	99.22	43.15	11	0.1757

TABLE 7. AMLI methods for Example 3: First choice of $T_{l_{k+1}}^{p,r}$ given in (57) with C^{p-1} regularity

$1/h$	t_c	t_s		n_{it}		ρ	
		L1	N2	L1	N2	L1	N2
$p=2$							
4	0.00	0.00	0.00	8	8	0.0899	0.0899
8	0.04	0.01	0.01	12	10	0.1913	0.1438
16	0.60	0.10	0.10	13	10	0.2400	0.1484
32	7.18	1.09	0.89	15	10	0.2694	0.1346
64	*	*	*	15	9	0.2830	0.1168
$p=3$							
4	0.00	0.00	0.00	10	10	0.1415	0.1415
8	0.15	0.02	0.03	14	13	0.2492	0.2304
16	2.84	0.27	0.24	15	11	0.2923	0.1862
32	35.61	2.79	2.21	17	11	0.3215	0.1762
64	*	*	*	17	11	0.3349	0.1738
$p=4$							
4	0.01	0.01	0.01	10	10	0.1443	0.1443
8	0.52	0.06	0.07	16	16	0.3027	0.3040
16	14.81	0.82	0.85	20	17	0.3900	0.3324
32	213.74	8.82	7.55	21	15	0.4067	0.2927
64	*	*	*	21	14	0.4042	0.2546

TABLE 8. AMLI methods for Example 3: with C^0 regularity

with first choice of $T_{l_{k+1}}^{p,r}$ given in (59)					with second choice of $T_{l_{k+1}}^{p,r}$ given in (60)				
$1/h$	t_c	t_s	n_{it}	ρ	$1/h$	t_c	t_s	n_{it}	ρ
		N2	N2(N3)	N2			N2	N2	N2
$p=2$					$p=2$				
4	0.01	0.01	12 (12)	0.2124	4	0.37	0.31	11	0.1753
8	0.11	0.05	15 (15)	0.2904	8	0.32	0.13	13	0.2212
16	1.25	0.52	16 (15)	0.2996	16	4.29	0.76	13	0.2250
32	12.06	4.51	16 (15)	0.3022	32	33.13	7.44	13	0.2261
$p=3$					$p=3$				
4	0.07	0.04	18 (18)	0.3527	4	0.09	0.03	14	0.2663
8	1.09	0.50	23 (22)	0.4408	8	1.42	0.34	16	0.3092
16	12.23	5.13	26 (23)	0.4919	16	15.72	3.30	17	0.3342
32	114.77	49.48	28 (23)	0.5164	32	123.05	32.24	18	0.3415
$p=4$					$p=4$				
4	0.39	0.45	48 (48)	0.6770	4	0.98	0.23	16	0.2987
8	5.84	4.36	54 (50)	0.7081	8	13.39	1.84	18	0.3465
16	64.09	47.27	64 (51)	0.7497	16	144.03	17.32	18	0.3560
32	*	*	73 (51)	0.7764	32	*	*	18	0.3577

rigorous local analysis of the CBS constant γ, particularly due to the overlapped support of B-splines, is not a straight forward task, and is still an open problem. We intend to address this issue in our future work.

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